$$W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \cdots & \cdots\cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

2.13. The existence and uniqueness of solution of linear equation (1) discussed in Art. 2.7. On an interval $a \le x \le b$, let the functions $p_0, p_1, \ldots, p_n$ and $r(x)$ be real and continuous, with $p_0(x) \ne 0$. Let x_0 be any point of the interval and let $k_0, k_1, \ldots k_{n-1}$ be any n real constants. Then there exists a unique solution of the differential equation $L(y) = r(x)$ on the interval that satisfies the initial conditions

$$y(x_0) = k_0, y'(x_0) = k_1, \ldots, y^{(k-1)}(x_0) = k_{n-1}.$$

The above theorem is an existence theorem because it says that the initial value problem does have a solution. It is also a uniqueness theorem, because it says that there is only one solution. Clearly, this theorem also applies to a homogeneous equation.

Note. In this chapter we shall assume without proof the above basic theorem for initial value problems associated with linear differential equations.

2.14. Some Useful Theorems

Theorem I. *A linear homogeneous equation of order n has not more than n linearly independent solutions.*

Proof. Let $u_1, u_2, \ldots, u_m$ be solutions of linear equation

$$p_0(x)y^{(n)} + \ldots + p_n(x)y = 0, \qquad \ldots(1)$$

where $m > n$. Let x_0 be any point of $a \le x \le b$. Then consider the following n equations in m unknowns $c_1, \ldots, c_m$.

$$\left.\begin{aligned} c_1u_1(x_0) + \ldots + c_mu_m(x_0) &= 0 \\ c_1u_1'(x_0) + \ldots + c_mu_m'(x_0) &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \text{and} \quad c_1u_1^{(n-1)}(x_0) + \ldots + c_mu_m^{(n-1)}(x_0) &= 0 \end{aligned}\right\} \qquad \ldots(2)$$

We know that a system of linear homogeneous equations has a non trivial solution if the number of equations is less than the number of unknowns.

Here $m > n$. Hence there exists a non-trivial solution for the system (2). Corresponding to such a non-trivial solution, we write

$$v(x) = c_1u_1(x) + \ldots + c_mu_m(x). \qquad \ldots(3)$$

Since, $u_1, u_2, \ldots, u_n$ are solutions of (1), $v(x)$ will also satisfy (1). Replacing x by x_0 in (3) and using first equation of (2), we get $v(x_0) = 0$. Now differentiating (3), we get

$$v'(x) = c_1u_1'(x) + \ldots + c_mu_m'(x). \qquad \ldots(4)$$

Replacing x be x_0 and making use of the second equation of (2), we get

$v'(x_0) = 0$. Again differentiate (4), use the third equation of (2) and get similarly $v''(x_0) = 0$. Proceeding likewise we shall get

$$v(x_0) = v'(x_0) = \ldots = v^{(n-1)}(x_0) = 0.$$

But $y = 0$ satisfies (1) and vanishes with all its derivatives at x_0. Consequently by the uniqueness theorem, $v(x) \equiv 0$ for $a \le x \le b$. Putting this value of $v(x)$ in (4), we get

$$c_1u_1 + c_2u_2 + \ldots + c_mu_m \equiv 0. \qquad \ldots(5)$$

where $c_1, c_2, \ldots, c_m$ are constants, not all zero. The equation (5) shows that $u_1, u_2, \ldots, u_m$ are dependent solutions of (1). This proves the required result.

Theorem II. *Let $u_1, u_2, \ldots, u_n$ be a linearly dependent set of functions on $a \le x \le b$, and let each function be $(n-1)$ times differentiable in (a, b). Then the Wronskian of the set of functions is identically zero.*

Proof. Since, by hyphothesis, the functions are linearly dependent on $a \le x \le b$, there must exist constants $c_1, \ldots, c_n$, not all zero, such that

$$c_1u_1(x) + \ldots + c_nu_n(x) \equiv 0$$

on $a \le x \le b$. Since the given functions are differentiable $(n-1)$ times, we defferentiate $(n-1)$ times and finaly obtain the n relations

$$\ldots(1) \qquad \left.\begin{aligned} c_1u_1(x) + \ldots + c_nu_n(x) &\equiv 0 \\ c_1u_1'(x) + \ldots + c_nu_n'(x) &\equiv 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \text{and} \quad c_1u_1^{(n-1)}(x) + \ldots + c_nu_n^{(n-1)}(x) &\equiv 0. \end{aligned}\right\}$$

If we consider a particular value $x = x_0$ in (a, b) then (1) represents a system of n homogeneous equations in n unknowns $c_1, c_2, \ldots, c_n$. Since not all c's are zero, the system (1) has a non-trivial solution. But this can happen only if the determinant of the system vanishes, that is,

$$W(u_1, u_2, \ldots, u_n)(x_0) = 0.$$

But x_0 is an arbitrary point of $a \le x \le b$, so

$$W(u_1, u_2, \ldots, y_n)(x) \equiv 0 \text{ on } a \le x \le b.$$

Corollary to Theorem II. *If the Wronskian of a set of functions is not zero, even at one point of the interval $a \le x \le b$, then the functions are linearly independent on $a \le x \le b$.*

Proof. Left to the reader.

Theorem III. *If $u_1, \ldots, u_n$ be n solutions of a linear homogeneous equation in the finite interval (a, b) whose Wronskian vanishes at any point (a, b), then these solutions are linearly dependent.*

Proof. $\qquad L(y) = y^{(n)} + p_1y^{(n-1)} + \ldots + p_{n-1}y = 0 \quad \ldots(1)$

is given differential equation. Let $u_1, \ldots, u_n$ be n solutions of (1).

Let us consider the system of n homogeneous equations in n unknowns $c_1, c_2, \ldots, c_n$ given below :

$$\left.\begin{aligned} c_1u_1(x_0)+\ldots+c_nu_n(x_0) &= 0 \\ c_1u_1'(x_0)+\ldots+c_nu_n'(x_0) &= 0, \\ \ldots\ldots\ldots\ldots\ldots &\ \ldots \ldots \\ \text{and} \quad c_1u_1^{(n-1)}(x_0)+\ldots+c_nu_n^{(n-1)}(x_0) &= 0. \end{aligned}\right\} \quad \ldots(2)$$

Now the determinant of the system (2) is $W(u_1, \ldots, u_n)(x_0)$, which vanishes by hypothesis, hence there must exist a non-trivial solution of the system (2). Corresponding to such a non-trivial solution, we write

$$v(x) = c_1u_1(x)+\ldots+c_nu_n(x) \quad \ldots(3)$$

wherein $c_1, \ldots, c_n$ are constants which are not all zero.

Since $u_1, \ldots, u_n$ are solutions of $L(y)=0$, $v(x)$ will also be solution of $L(y) = 0$. Replacing x by x_0 in (3) and using first equation of (2), we get $v(x_0)=0$. Now differentiating (3), we get

$$v'(x) = c_1u_1'(x)+\ldots+c_nu_n'(x). \quad \ldots(4)$$

Replacing x by x_0 and using the second equation of (2), we get $v'(x_0)=0$. Again differentiate (4) and use the third equation of (2), then we get $v''(x_0)=0$. In this manner we get

$$v(x_0) = v'(x_0) = \ldots = v^{(n-1)}(x_0) = 0.$$

But $y = 0$ satisfies (1) and vanishes with all its derivatives at x_0. Consequently by the uniqueness theorem, $v(x) \equiv 0$ for $a \le x \le b$. Using this value of $v(x)$, (3) reduces to

$$c_1u_1+\ldots+c_nu_n \equiv 0$$

on $a \le x \le b$. Since the constants $c_1, \ldots, c_n$ are not all zero, it follows that the function $u_1, \ldots, c_n$ are linearly dependent on $a \le x \le b$.

Theorem IV. *Let the functions $u_1, \ldots, u_n$ be solutions of an nth-order linear homogeneous equation on $a \le x \le b$. Then either the Wronskian of these functions is identically zero on $a \le x \le b$ (in which case the functions are linearly dependent) or it does not vanish at any point $a \le x \le b$ (in which case the functions are linearly independent).*

Proof. We know that the functions $u_1, \ldots, u_n$ are either linearly dependent or linearly independent. If they are dependent, their Wronskian vanishes identically on $a \le x \le b$, by theorem II. On the other hand, if the functions are independent, their Wronskian cannot vanish at any point of $a \le x \le b$. For if it did, the functions would be linearly dependent by virtue of theorem III.

Note. For complete proof of theorem IV, the proofs of both the theorems II and III must be given.

Theorem V. *The necessary and sufficient condition that the n integrals $y_1, y_2, \ldots, y_n$ of the linear differential equation*

$$(d^ny/dx^n)+P_1(d^{n-1}y/dx^{n-1})+\ldots+P_ny=0,$$

where $P_1, P_2, \ldots, P_n$ are functions of x or constants, are linearly independent is that the determinant

$$\begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

does not vanish identically. **[Meerut 92, 93, 96, 97]**

Proof. Let the given determinant be denoted by $W(x)$. Then

$$W(x) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}. \quad \ldots(1)$$

The condition is sufficient. Let $W(x) \neq 0$. Then we are to show that $y_1, y_2, \ldots, y_n$ are linearly independent.

Let $y_1, y_2, \ldots, y_n$ be not linearly independent *i.e.*, $y_1, y_2, \ldots, y_n$ be linearly dependent. Then, by definition, there must exist n constants $c_1, c_2, \ldots, c_n$, not all equal to zero, such that

$$a_1y_1+a_2y_2+\ldots+a_ny_n = 0. \quad \ldots(2)$$

Now from (2), we have

$$\left.\begin{aligned} a_1y_1'+a_2y_2'+\ldots+a_ny_n' &= 0, \\ \ldots\ldots\ldots\ldots\ldots &\ \ldots\ldots \\ \text{and} \quad a_1y_1^{(n-1)}+a_2y_2^{(n-1)}+\ldots+a_ny_n^{(n-1)} &= 0. \end{aligned}\right\} \quad \ldots(3)$$

Assuming that $a_1 \neq 0$, (1) can be rewritten as

$$W(x) = \frac{1}{a_1}\begin{vmatrix} a_1y_1 & y_2 & \ldots & y_n \\ a_1y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ a_1y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

$$= \frac{1}{a_1}\begin{vmatrix} a_1y_1+a_2y_2+\ldots+a_ny_n & y_2 & \ldots & y_n \\ a_1y_1'+a_2y_2'+\ldots+a_ny_n' & y_2' & \ldots & y_n' \\ \ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots \\ a_1y_1^{(n-1)}+a_2y_2^{(n-1)}+\ldots+a_ny_n^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

[Performing $C_1 \to C_1+a_2C_2+a_3C_3+\ldots+a_nC_n$]

$$= \frac{1}{a_1}\begin{vmatrix} 0 & y_2 & \ldots & y_n \\ 0 & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ 0 & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

$$= 0,$$

showing that if $y_1, y_2, \ldots, y_n$ are linearly dependent, then $W(x)$ will be identically equal to zero. It then follows that if $W(x) \neq 0$, then $y_1, y_2, \ldots, y_n$ must be linearly independent.

The condition is necessary. Let $y_1, y_2, \ldots, y_n$ be linearly independent. Then we are to show that $W(x) \neq 0$.

Let $W(x) = 0$. Then we wish to prove that $y_1, y_2, ..., y_n$ are linearly dependent *i.e.*, there exist a relation of the form (2). We shall prove it by using mathematical induction on n number of functions. Suppose that the condition holds in the case of $(n-1)$ functions.

Since $W(x) = 0$, it follows that $W(x)$ must be reducible to a form wherein all the elements of one column (or row) are zero. Accordingly, there must exist certain multipliers $\lambda_1, \lambda_2, ..., \lambda_n$, such that

$$\left.\begin{aligned} \lambda_1 y_1 + \lambda_2 y_2 + ... + \lambda_n y_n &= 0 \\ \lambda_1 y_1' + \lambda_2 y_2' + ... + \lambda_n y_n' &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \text{and} \quad \lambda_1 y_1^{(n-1)} + \lambda_2 y_2^{(n-1)} + ... + \lambda_n y_n^{(n-1)} &= 0. \end{aligned}\right\} \quad ...(4)$$

Differentiating each of these equations and then substracting each from the next following equation, we get

$$\left.\begin{aligned} \lambda_1' y_1 + \lambda_2' y_2 + ... + \lambda_n' y_n &= 0 \\ \lambda_1' y_1' + \lambda_2' y_2' + ... + \lambda_n' y_n' &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \; \cdots \cdots \\ \text{and} \quad \lambda_1' y_1^{(n-2)} + \lambda_2' y_2^{(n-2)} + ... + \lambda_n' y_n^{(n-2)} &= 0. \end{aligned}\right\} \quad ...(5)$$

Consider the matrix, $M = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \end{bmatrix}$

Suppose that one of the determinants of order $n-1$ formed from the above matrix M vanishes, say the one formed by omitting the rth column, then by hypothesis there exist is a relation of the form

$$c_1 y_1 + c_2 y_2 + ... + c_{r-1} y_{r-1} + c_{r+1} y_{r+1} + ... + c_n y_n = 0.$$

On the other hand if none of the above mentioned type of determinants vanish, then from (5) and the first $(n-1)$ equations in (4), we have

$$\frac{\lambda_1'}{\lambda_1} = \frac{\lambda_2'}{\lambda_2} = ... = \frac{\lambda_n'}{\lambda_n} = \mu, \text{(say)} \quad ...(6)$$

$$\left.\begin{aligned} &\text{From (6), } \lambda_1' = \mu \lambda_1 \text{ or } \frac{d\lambda_1}{dx} = \mu \lambda_1 \text{ or } \frac{d\lambda_1}{\lambda_1} = \mu dx. \\ &\text{Integrating, } \log \lambda_1 - \log a_1 = \int \mu dx \text{ or } \lambda_1 = a_1 e^{\int \mu dx}. \end{aligned}\right\} (7)$$

Similarly, (6) gives $\lambda_2 = a_2 e^{\int \mu dx}, ..., \lambda_n = a_n e^{\int \mu dx}$.

Here $a_1, a_2, ..., a_n$ are the constants of integration. Putting the values of $\lambda_1, \lambda_2, ..., \lambda_n$ given by (7) in first of the equations (3) and dividing by $e^{\int \mu dx}$, we have

$$a_1 y_1 + a_2 y_2 + ... + a_n y_n = 0.$$

Thus, if the result holds for $n-1$ functions, it holds also for n functions.

We now prove that the result holds for one function. Now, determinant

$|y_1| = 0 \Rightarrow y_1 = 0$. Hence there exist a non-zero constant a_1 such that $a_1 y_1 = 0$.

Thus the result holds for one function. Also if the condition holds for $n-1$ functions, then the condition holds for n functions. Hence by mathematical induction the result is true for every positive integer n.

Thus $W(x) = 0 \Rightarrow y_1, y_2, ..., y_n$ are linearly dependent. It follows that if $y_1, y_2, ..., y_n$ are linearly independent, then $W(x)$ must vanish identically.

SOLVED EXAMPLES

Ex. 1. *If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of differential equation $y'' + 9y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.* **[Delhi B.Sc. (Hons) 1996]**

Sol. The Wronskian of $y_1(x)$ and $y_2(x)$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} = -3\sin^2 3x - 3\cos^2 3x$$

$$= -3(\sin^2 3x + \cos^2 3x) = -3 \neq 0.$$

Since $W(x) \neq 0$, $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y'' + 9y = 0$.

Ex. 2. *Prove that $\sin 2x$ and $\cos 2x$ are solutions of the differential equation $y'' + 4y = 0$ and these solutions are linearly independent.* **[Delhi (B.Sc.) (G) 1998]**

Sol. Given equation is $\quad y'' + 4y = 0.$ ...(1)

Let $\quad y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x.$ ...(2)

Now, $\quad y_1' = 2\cos 2x$ and $y_1'' = -4\sin 2x.$ ...(3)

$\therefore \quad y_1''(x) + 4y_1(x) = -4\sin 2x + 4\sin 2x = 0$, by (2) and (3)

Hence, $y_1(x) = \sin 2x$ is a solution of (1). Similarly we can prove that $y_2(x)$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2\sin^2 2x - 2\cos^2 2x$$

$$= -2(\sin^2 2x + \cos^2 2x) = -2 \neq 0.$$

Since $W(x) \neq 0$, $\sin 2x$ and $\cos 2x$ are linearly independent solutions of (1).

Ex. 3. *Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2$, $y'(0) = 3$.* **[Delhi B.Sc. (P) 96]**

Sol. Given equation is $\quad y'' - 2y' + 2y = 0.$ ...(1)

Let $\quad y_1(x) = e^x \sin x$ and $y_2(x) = e^x \cos x.$ ...(2)

From (2), $\quad y_1'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$ ...(3)

From (3), $\quad y_1''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x.$ ...(4)